On 'Rotating regular black hole solution': Generating the physical solution in Boyer-Lindquist coordinates

Mustapha Azreg-Aïnou
Başkent University, Department of Mathematics, Bağlıca Campus, Ankara, Turkey

We show that the rotating metric proposed in Phys. Rev. D **89**, 104017 (arXiv:1404.6443 [gr-qc]) does not result from application of Newman-Janis algorithm. Dropping the complexification procedure and introducing more physical arguments and symmetry properties, we show how one can remedy the situation by providing the appropriate, easy to handle, metric for generating, regular and singular, rotating black hole and non-black-hole solutions in Boyer-Lindquist coordinates. We focus on rotating regular black holes and show that they are regular on the Kerr-like ring but physical entities are undefined there. We show that rotating regular black holes have much smaller electric charges and, with increasing charge, they turn into regular non-black-hole solutions well before their Kerr-Newman counterparts become naked singularities. No causality violations occur in the region inside a rotating regular black hole.

I. ON THE NEWMAN-JANIS ALGORITHM

In this introductory section we comment on two steps in the Newman-Janis algorithm (NJA) [1]. We first introduce the following general static metric

$$\mathrm{d}s^2_{\text{stat}} = G(r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{F(r)} - H(r)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \quad (1)$$

One of the ambiguous steps in the algorithm is complexification of the radial coordinate r. This is the step that follows the complex coordinate transformation:

$$r \to r + \mathrm{i}a\cos\theta,\ u \to u - \mathrm{i}a\cos\theta,$$

where (u, r, θ, φ) are the advanced null coordinates. Recall that there were already generalizations of this complex coordinate transformation [2], but it seems that the subsequent developments of the NJA and generating methods have not made the matter of further generalizing these transformations a concern. There are as many ways to complexify r as one wants. Here are some examples:

$$\begin{aligned} &r^2 \to (r+\mathrm{i}a\cos\theta)(r-\mathrm{i}a\cos\theta) = r^2 + a^2\cos^2\theta,\\ &r \to \frac{2(r+\mathrm{i}a\cos\theta)(r-\mathrm{i}a\cos\theta)}{(r+\mathrm{i}a\cos\theta)+(r-\mathrm{i}a\cos\theta)} = \frac{r^2+a^2\cos^2\theta}{r}, \quad (2)\\ &r^2 \to r\sqrt{(r+\mathrm{i}a\cos\theta)(r-\mathrm{i}a\cos\theta)} = r\sqrt{r^2+a^2\cos^2\theta}. \end{aligned}$$

When $a = 0$, each r.h.s reduces to the left hand side (l.h.s) of the same line. Both the first and second types of complexification in (2) are used to derive the Kerr solution from the Schwarzschild one: If only one type of complexification is used, the generated rotating solution will not look like the Kerr one! This is the very ambiguity behind nonphysical solutions [3] that cannot be written in Boyer-Lindquist coordinates (BLC's) as shown in [4].

The failure of the last step of NJA, which consists in bringing the generated rotating solution written in Eddington-Finkelstein coordinates (EFC's) to BLC's by real coordinate transformation, is likely related to the complexification procedure. We have already commented on this point in [4] and have shown that it is not possible in general to carry this last step of NJA. In this work we will rise similar comments concerning the results derived in [5].

The issue pertaining to complexification has been solved in [6] where a generic metric formula, not appealing to complexification procedure, was derived to generate imperfect fluid rotating solutions in BLC's. The metric formula depends on a three-variable function $\Psi(r, \theta, a)$ whose determination depends on the physical problem at hands, that is, it depends on the type of rotating solution one wants to derive. Ψ generally obeys some partial differential equation(s). In the case to which one is generally interested, where the source term in the field equations, $T^{\mu\nu}$, is interpreted as an imperfect fluid rotating about a fixed axis, Ψ obeys two linear and nonlinear partial differential equations [6, Eqs.(15),(18)] and [7, Eqs.(4),(7)]. Thus, the essence of our procedure is to reduce the task of determining the rotating counterpart of (1) to that of fixing Ψ by solving nonlinear partial differential equations. Applications are considered in [6, 7] and in Sect. III of this work.

In Sect. II we comment on Ref. [5] and show that the rotating metric derived there by NJA cannot be brought to BLC's. In Sect. III we derive, based on our previous works [6, 7], a rotating regular black hole in BLC's. Sect. IV is devoted to the discussion of the general properties of all rotating regular black holes as well as to the special properties of the rotating regular black hole counterpart of the static regular one derived in [8]. We conclude in Sect. V. An appendix has been added to derive the extremality condition of the rotating regular black hole.

II. COMMENT ON REF. [5]

Eqs. (2.2) and (2.15) of Ref. [5] reveal that both the first and third types of complexification in (2) have been used to derive the "rotating" solution (2.19) of Ref. [5] in

EFC's. No explication of why these two complexifications have been selected occurs in Ref. [5].

In Eqs. (2.20) of Ref. [5], each r.h.s is a total differential (exact differential) provided the functions λ and χ depend only on r. It is easy to check that, in this case, the conditions of integrability are satisfied, so one can integrate the two equations to obtain global coordinates $u(t,r)$ and $\phi(\phi,r)$.

Unfortunately, this is not the case in the final expressions of λ and χ given in the right hand sides (r.h.s) of Eqs. (2.21) of Ref. [5], which generally depend on both (r,θ): Only in the trivial case $Q=0$, which corresponds to Schwarzschild solution, λ and χ depend on r only.

If λ and χ depend on both (r,θ) then $\partial\lambda/\partial\theta \neq 0$ and $\partial\chi/\partial\theta \neq 0$, so the conditions of integrability are no longer satisfied and it is not possible to integrate Eqs. (2.20) of Ref. [5] to obtain global coordinates $u(t,r,\theta)$ and $\phi(\phi,r,\theta)$. Consequently, if $Q \neq 0$, the set of Eqs. (2.20) of Ref. [5] does not constitute a coordinate transformation and the final metric, Eqs. (2.22) of Ref. [5], is not equivalent to the metric (2.19) of Ref. [5]. Said otherwise, if $Q \neq 0$, it is not possible, by a coordinate transformation, to bring metric (2.19) to metric (2.22) of Ref. [5].

Thus, metric (2.22) of Ref. [5] does not result from application of NJA and there is no remedy to help overcome the situation[1]. One may think, for instance, to postulate the metric independently of NJA. If that were the case one would again encounter the complexification problem because the function $\widetilde{f}(r)$ is nothing but $f(r)$ where r has been subject to complexification. Second, one may modify some metric components in Eq. (2.22) of Ref. [5] without modifying its asymptotic behavior nor its behavior in the limit $a \to 0$: For instance, multiply the coefficient of $\mathrm{d}t^2$ by $(r^2+a^2)/(r^2+a^2\cos^2\theta)$ (or by its inverse) to get a new metric postulate with similar spatial asymptotic properties and behavior as $a \to 0$. How many are there such metric postulates derived this way without using physical arguments? And which metric postulate yields a physical solution?

In the following section we show how one can skip the complexification procedure and we introduce more physical arguments and symmetry properties to derive rotating regular black hole counterparts of static regular ones. We apply the rules to the static solution used in Ref. [5] and derive a rotating regular solution for it that is much more simpler than the one "postulated" in Ref. [5].

[1] For the same reason, because of its non-simple structure the metric (3.12) of [9] [Eq. (18) of arXiv:1005.5605] does not seem to result from application of NJA as claimed.

III. A ROTATING REGULAR BLACK HOLE

In [6] we dropped the complexification procedure and obtained a metric in EFC's depending on three unknown functions $(A(r,\theta,a), B(r,\theta,a), \Psi(r,\theta,a))$ and on (G,F,H). We then performed a coordinate transformation on the rotating metric in EFC's with well defined functions $\lambda(r)$ and $\chi(r)$ and we required the final transformed metric be brought to BLC's. This fixed two of the three unknown functions, (A,B), and resulted in a metric formula depending on the three-variable function $\Psi(r,\theta,a)$ to be fixed using physical arguments.

From now on, we will use the notation of [6, 7]. Let

$$K(r) \equiv \sqrt{F}H/\sqrt{G}, \tag{3}$$

where (G,F,H) is the static metric (1). Its rotating counterpart in BLC's takes the form [6, 7]

$$\begin{aligned}\mathrm{d}s^2 =& \frac{(FH+a^2\cos^2\theta)\Psi\mathrm{d}t^2}{(K+a^2\cos^2\theta)^2} - \frac{\Psi\mathrm{d}r^2}{FH+a^2}\\ &+ 2a\sin^2\theta\Big[\frac{K-FH}{(K+a^2\cos^2\theta)^2}\Big]\Psi\mathrm{d}t\mathrm{d}\phi - \Psi\mathrm{d}\theta^2\\ &- \Psi\sin^2\theta\Big[1+a^2\sin^2\theta\frac{2K-FH+a^2\cos^2\theta}{(K+a^2\cos^2\theta)^2}\Big]\mathrm{d}\phi^2, \end{aligned} \tag{4}$$

and is brought to Kerr-like forms

$$\begin{aligned}\mathrm{d}s^2 =& \frac{\Psi}{\rho^2}\Big[\Big(1-\frac{2f}{\rho^2}\Big)\mathrm{d}t^2 - \frac{\rho^2}{\Delta}\,\mathrm{d}r^2\\ &+ \frac{4af\sin^2\theta}{\rho^2}\,\mathrm{d}t\mathrm{d}\phi - \rho^2\mathrm{d}\theta^2 - \frac{\Sigma\sin^2\theta}{\rho^2}\,\mathrm{d}\phi^2\Big]\end{aligned} \tag{5}$$

$$\begin{aligned}\mathrm{d}s^2 =& \frac{\Psi}{\rho^2}\Big[\frac{\Delta}{\rho^2}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)^2 - \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2\\ &- \frac{\sin^2\theta}{\rho^2}\,[a\mathrm{d}t-(K+a^2)\mathrm{d}\phi]^2\Big].\end{aligned} \tag{6}$$

on performing the following variable changes:

$$\begin{aligned}&\rho^2 \equiv K + a^2\cos^2\theta,\;\; 2f(r) \equiv K - FH\\ &\Delta(r) \equiv FH + a^2,\;\; \Sigma \equiv (K+a^2)^2 - a^2\Delta\sin^2\theta.\end{aligned} \tag{7}$$

Here $\Psi(r,\theta,a)$ is an unknown function. If the source term $T^{\mu\nu}$ is interpreted as an imperfect fluid rotating about the z axis, Ψ obeys the two linear and nonlinear partial differential equations (15) and (18) of [6] to which some particular solutions were found in [6, 7]. The nonlinear differential equation is just $G_{r\theta}=0$, where $G_{\mu\nu}$ is the Einstein tensor, and the linear differential equation ensures consistency of the field equations, $G_{\mu\nu}=T_{\mu\nu}$, with the expression of $T^{\mu\nu}$: $T^{\mu\nu} = \epsilon e^\mu_t e^\nu_t + p_r e^\mu_r e^\nu_r + p_\theta e^\mu_\theta e^\nu_\theta + p_\phi e^\mu_\phi e^\nu_\phi$, where e^μ_t is the four-velocity vector of the fluid, ϵ is the density, $(p_r,\, p_\theta,\, p_\phi)$ are the components of the pressure, and the basis $(e_t,\, e_r,\, e_\theta,\, e_\phi)$ is dual to the 1-forms defined in (6): $\omega^t \equiv \sqrt{\Psi\Delta}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)/\rho^2$,

$\omega^r \equiv -\sqrt{\Psi}\mathrm{d}r/\sqrt{\Delta}$, $\omega^\theta \equiv -\sqrt{\Psi}\mathrm{d}\theta$, $\omega^\phi \equiv -\sqrt{\Psi}\sin\theta[a\mathrm{d}t-(K+a^2)\mathrm{d}\phi]/\rho^2$ [6, Eqs.(16),(17)].

In our notation, the static metric (2.1) of Ref. [5], which was derived in [8], takes the form:

$$G = F = 1 - \frac{2Mr^2}{(r^2+Q^2)^{3/2}} + \frac{Q^2r^2}{(r^2+Q^2)^2}, \quad H = r^2, \tag{8}$$

which implies $K = r^2$. Among the solutions known for Ψ, we are interested to the special solution (19) of [6], which is valid in our case where $K = r^2$ provided we take $q^2 = 0$:

$$\Psi = r^2 + a^2\cos^2\theta. \tag{9}$$

In the case $G = F$, a general prescription for determining imperfect fluid rotating (about the z axis) regular black holes is given in [6, Sect. 3]. Using (8), (9), and (7) along with $\rho^2 = r^2 + a^2\cos^2\theta = \Psi$ in (5), the regular rotating counterpart black hole of (8) takes the compact form:

$$\begin{aligned}
\mathrm{d}s^2 &= \Big(1-\frac{2f}{\rho^2}\Big)\mathrm{d}t^2 - \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 \\
&+ \frac{4af\sin^2\theta}{\rho^2}\,\mathrm{d}t\mathrm{d}\phi - \rho^2\mathrm{d}\theta^2 - \frac{\Sigma\sin^2\theta}{\rho^2}\,\mathrm{d}\phi^2 \qquad (10)\\
\rho^2 &= r^2 + a^2\cos^2\theta, \;\; 2f = r^2(1-F) \\
\Delta &= r^2F + a^2, \;\; \Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta,
\end{aligned}$$

which reduces to Kerr metric if $Q = 0$ where, in this case, $2f = 2Mr$, $\Delta = r^2 - 2Mr + a^2$, and $\Sigma = (r^2+a^2)\rho^2 + 2Ma^2r\sin^2\theta$.

IV. PHYSICAL PROPERTIES

In this section we discuss the general and special properties of (10) for rotating regular black holes as well as for singular ones that can be generated from a static regular or singular metric. However, we focus mostly on rotating regular black holes. The first part of this section is devoted to a general discussion and the second one is concerned with the special solution (10) where F is given by (8).

A. General physical properties

Notice that the only difference between Kerr's metric and (10) resides in the values of (f, Δ, Σ). Moreover, and this is an important point in our method, metric (10) is a fresh formula, that is, it applies to all static (being regular or not) black holes of the form (1) with $G = F$ and $H = r^2$, the only task one has to perform is to evaluate $2f = r^2(1-F)$, $\Delta = r^2F + a^2$, and $\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$ knowing F. There are no notions of complexification associated with the different forms of F while the application of NJA necessitates different ways of complexification for each different form of F and the final rotating metric may only be given in EFC's because of nonexistence of coordinate transformations bringing it to BLC's, as were the cases in [3, 5].

Stress-energy tensor. We keep on doing generalities which apply to all rotating regular black holes of the form (10) (other conclusions apply also to singular solutions). The components $(\epsilon, p_r, p_\theta, p_\phi)$ of the stress-energy tensor (SET), $T^{\mu\nu}$, are given by Eqs. (13) and (14) of [7] taking $p^2 = 0$ (these have been evaluated in [10, 11] too):

$$\epsilon = -p_r = \frac{2(rf'-f)}{\rho^4}, \quad p_\theta = p_\phi = -p_r - \frac{f''}{\rho^2}, \tag{11}$$

(here $f' \equiv \mathrm{d}f/\mathrm{d}r$) which, despite their appearance, have been shown to remain finite, but undefined, on the ring $\rho^2 = 0$ because of de Sitter-like behavior near $r = 0$ of the static regular black hole ($F \sim 1-\text{const}\,r^2$ and const > 0), as is the case with all static regular holes (see paragraph following Eq. (14) of [7] and Case (1) of [6]). The curvature scalar remains finite too and undefined on the ring (see Case (1) of [6] for a general discussion). Because of the relation $p_r = -\epsilon$, these solutions can also be used as regular cores to match other rotating external solutions [7]. Note that NJA was first devised to generate exterior rotating solutions but later was applied to generate rotating interior metrics which were matched to the exterior Kerr one [12, 13].

Notice from (11) that, since f does not depend on the rotation parameter a, ϵ has the same sign as its static counterpart ϵ_{stat}: $\epsilon = (r^4/\rho^4)\epsilon_{\text{stat}}$. This remark is very relevant for the determination of the energy conditions of rotating regular black holes. For the rotating regular black hole solution (10) with F given by (8), it was reported that its static counterpart black hole satisfies the weak energy condition [14], that is, $\epsilon_{\text{stat}} \geq 0$, we thus conclude that $\epsilon \geq 0$. Because of de Sitter-like behavior near $r = 0$ of the static regular black hole, this latter conclusion is valid for all rotating regular black holes near $r = 0$ where $rf' - f \simeq 3\text{const}\,r^4$.

It is straightforward to check that the components of the SET given by (11) approach those of Kerr-Newman black hole in the limit $r \to \infty$ if F approaches the Reissner-Nordström limit.

The function f'' is zero only for Reissner-Nordström-like solutions of the form $F = 1 + c_1/r + c_2/r^2$. For all other, regular or singular, solutions $f'' \neq 0$ and, by (11), $p_\theta = p_\phi \neq \epsilon = -p_r$, so the fluid is never perfect.

Static limit – Horizons. The mass of the rotating solution, being regular or not, is that of the static one. This is obvious from (10) for if F admits a Taylor expansion of the form $F = 1 - 2M/r + \cdots$ at spatial infinity, then the two metric functions $g_{tt} =$ and $1/g_{rr}$ of the rotating solution (10) admit the same expansion as $r \to \infty$.

The static limit, which is the 2-surface on which the timelike Killing vector $t^\mu = (1, 0, 0, 0)$ becomes null, corresponds to $g_{tt}(r_{\rm st},\theta)=0$ leading to $2f=\rho^2$ or simply

$$a^2\cos^2\theta=-{r_{\rm st}}^2F(r_{\rm st}). \qquad (12)$$

Observers can remain static only for $r>r_{\rm st}(\theta)$. Notice that if $g_{tt}(r,\theta)$ were the metric "postulated" in [5, Eq.(2.22)], the corresponding equation $g_{tt}(r_{\rm st},\theta)=0$ would not be separable as in (12).

The event horizon r_+, which sets a limit for stationary observers, and the inner apparent one r_- are solutions to $g^{rr}(r_\pm)=0$ implying $\Delta(r_\pm)=0$:

$${r_\pm}^2F(r_\pm)+a^2=0. \qquad (13)$$

It is clear from these last two equations that the static limit and event horizon intersect only at the two poles $\theta=0$ and $\theta=\pi$ where $r_{\rm st}=r_+$, as in Kerr and Kerr-Newman solutions. The resolution of (13) provides $r_\pm$ as functions of the charges, $(M,Q,\cdots)$, on which F depends and a^2 only, contrary to the metric of [5, Eq.(2.22)] where $r_\pm$ are functions of θ too.

It is well known that, if $Q^2<M^2$, a Kerr-Newman solution may have the properties of a rotating black hole; this happens if $0<a^2\leq M^2-Q^2$, otherwise $(a^2>M^2-Q^2)$ the solution is a naked singularity. In the case where $Q^2\geq M^2$, a Kerr-Newman solution is always a naked singularity for all $a^2>0$. As we shall see in the next section, even in the case where $Q^2<M^2$, it is possible to have no rotating regular black holes for all a^2 but only regular non-black-hole solutions given by (10), as is the case shown in Fig. 1 (a) which is a plot of the extremality condition in the $(a^2/Q^2,M^2/Q^2)$ plane. Similar conclusion was made in [14]. If the function $\mathcal{F}(r)\equiv r^2F\,(=\Delta-a^2)$, which is zero at $r=0$ for a static regular black hole (resp. constant for a singular black hole) and $\mathcal{F}\to\infty$ as $r\to\infty$, has some negative minimum value $\mathcal{F}_{\rm min}$ on the range of r, then there is always a black hole solution if

$$0<a^2\leq|\mathcal{F}_{\rm min}| \qquad (14)$$

and a non-black-hole solution (resp. a naked singularity) for

$$a^2>|\mathcal{F}_{\rm min}|. \qquad (15)$$

The extremality condition is

$$a^2=|\mathcal{F}_{\rm min}| \qquad (16)$$

which provides a relation between the charges $(M,Q,\cdots)$ and a^2.

Causality issues. It is also well known that causality violations occur in Kerr and Kerr-Newman black holes, as depicted in Fig. 2 (a). Causality violations and closed timelike curves (CTCs) are possible if, in (10), $g_{\phi\phi}=-\Sigma\sin^2\theta/\rho^2>0$. Since $\sin^2\theta/\rho^2$ is not negative, for simplicity we investigate the sign of $\Sigma=(r^2+a^2)^2-a^2(r^2F+a^2)\sin^2\theta$. Fig. 2 (a) is a plot of r versus $\sin\theta$ where, for a given θ, r is a solution to $\Sigma(\sin\theta,r)=0$ and Fig. 2 (b) is a plot of r^2 versus $\sin\theta$ where r^2 is a solution to $\Sigma(\sin\theta,r^2)=0$. Causality violations occur on the right of each plot in Fig. 2 (a) where the dashed curve corresponds to Kerr black hole and the continuous one corresponds to Kerr-Newman black hole for which CTCs exist even for $r>0$ [in contrast with the Kerr hole where CTCs are possible for $r<0$ only, as depicted in Fig. 2 (a)]. In Fig. 2 (a), the plot of $\Sigma=0$ for the rotating regular hole (10) where F is given by (8) is the point $\sin\theta=1$ and $r=0$. Since for $\sin\theta=0$, $\Sigma>0$, this implies that $\Sigma\geq0$ at least for the values of the parameters we have chosen $M^2=16$, $Q^2=1$, and $a^2=1$ corresponding, according to Fig. 1 (a), to the black hole region for the rotating regular black hole solution (10) with F given by (8). This shows that there are no causality violations for this black hole since the sign of $g_{\phi\phi}$ can't go positive, that is, the Killing vector $\phi^\mu=(0,0,0,1)$, of norm $g_{\phi\phi}$, can't become timelike.

Let us see under which general conditions the above conclusion remains valid. Notice that causality violations are not expected in the region $r>r_+$ nor in in the region between the horizons since there $\Delta<0$ yielding $\Sigma>0$ and $g_{\phi\phi}<0$. Let $r<r_-$. The condition $\Sigma>0$ yields $(r^2+a^2)^2>a^2(r^2F+a^2)\sin^2\theta$. Since $\Delta=r^2F+a^2>0$ for $r<r_-$, if we can show that

$$(r^2+a^2)^2>a^2(r^2F+a^2), \qquad (17)$$

this results in $\Sigma>0$. Simplifying (17), we bring it to

$$r^2-a^2F(r)+2a^2>0. \qquad (18)$$

The condition (18) is satisfied at $r=0$ and $r=r_-$ where its l.h.s is a^2 and ${r_-}^2+(a^4/{r_-}^2)+2a^2$, respectively. Here we have used $F(0)=1$ and $\Delta(r_-)={r_-}^2F(r_-)+a^2=0$. Thus, if $r=\epsilon a$ or $r=r_--\eta$ where ϵ is a small positive or negative number[2] and η is a small positive number, there are no causality violations for all rotating regular black holes.

It might be true that the condition (18) holds for all $r<r_-$ including negative values down to $-r_-$. The derivative of the l.h.s of (18) is

$$2r-a^2F' \qquad (19)$$

which vanishes at $r=0$. Because of the Sitter behavior, the function F approaches 1 from below resulting in $F'<0$ near the origin. If $F'<0$ holds for all $0<r<r_-$, then $2r-a^2F'>0$ and the l.h.s of (18) increases from a^2 to ${r_-}^2+(a^4/{r_-}^2)+2a^2$, hence no causality violations occur for $0<r<r_-$. Even if $F'<0$ fails to be true for all $0<r<r_-$, the condition (18) may still hold unless F oscillates rapidly in the region $0<r<r_-$, in which case this would lead to a nonphysical solution.

[2] This same result could be achieved setting $r=\epsilon a$ and $\theta=(\pi/2)+\delta$, where δ is a small positive or negative number, yielding $\Sigma\simeq a^4(\epsilon^2+\delta^2)$.

Angular velocities. The angular velocity Ω of the rotating hole (10) is[3] $\Omega \equiv -g_{t\phi}/g_{\phi\phi} = 2af/\Sigma$: This is the angular velocity, attributable to dragging effects, of freely falling particles initially at rest at spatial infinity as they reach the point (r, θ). As $r \to \infty$, $\Omega \to 2Jr^{-3}$ where $J = Ma$ is the angular momentum of the rotating hole. The angular velocity of the horizon Ω_H is taken as $\Omega(r_+)$. Using $\Sigma(r_+) = ({r_+}^2 + a^2)^2$ along with (13), we obtain

$$\Omega_H = \frac{2af(r_+)}{\Sigma(r_+)} = \frac{a{r_+}^2[1 - F(r_+)]}{({r_+}^2 + a^2)^2} = \frac{a}{{r_+}^2 + a^2}. \quad (20)$$

The four-velocity of the fluid elements is $e^\mu_t = (r^2 + a^2, 0, 0, a)/\sqrt{\rho^2\Delta}$ [6, 7]. This can be written as $e^\mu_t = N(t^\mu + \omega\phi^\mu)$, in terms of the timelike $t^\mu = (1, 0, 0, 0)$ and spacelike $\phi^\mu = (0, 0, 0, 1)$ Killing vectors, with $N = (r^2 + a^2)/\sqrt{\rho^2\Delta}$ and $\omega = a/(r^2 + a^2)$ is the differentiable angular velocity of the fluid. Since the norm of the vector $t^\mu + \omega\phi^\mu$, $1/N^2$, is positive only for $\Delta > 0$, which corresponds to the region $r > r_+$, the fluid elements follow timelike world lines only for $r > r_+$. As $r \to r_+$, ω approaches the limit $a/({r_+}^2 + a^2)$ that is the largest angular velocity of the fluid elements and this equals the angular velocity of the event horizon (20). So the fluid elements are dragged with the angular velocity Ω_H as all falling objects. At the event horizon, $t^\mu + \omega\phi^\mu$ becomes null and tangent to the horizon's null generators.

parametric plot of $1/(2s)^2$ vs. u^2 and that of Fig. 1 (b) is a parametric plot of $t - 1$ vs. u^2

B. Special properties

We specialize to the case where F is given by (8). Eq. (13) takes to form where we drop the subscripts $\pm$:

$$r^2 - \frac{2Mr^4}{(r^2 + Q^2)^{3/2}} + \frac{Q^2r^4}{(r^2 + Q^2)^2} + a^2 = 0. \quad (21)$$

As we noticed earlier, the locations of the horizons are functions of (M, Q, a) only, contrary to the solution of [5, Eq.(2.22)] where these locations are functions of θ too. Unfortunately, one cannot solve (21) for r in terms of (M, Q, a). For $Q^2/M^2 \ll 1$, we obtain

$$r_\pm \simeq r_{\text{K}\,\pm} + c_\pm Q^2 \quad (22)$$
$$c_\pm = \frac{4M \pm \sqrt{M^2 - a^2}}{2[a^2 - M(M \pm \sqrt{M^2 - a^2})]},\ c_+ < 0,\ c_- > 0$$

where $r_{\text{K}\,\pm} = M \pm \sqrt{M^2 - a^2}$ are the horizons of the Kerr black hole. If $r_{\text{KN}\,\pm}$ denote the corresponding horizons of the Kerr-Newman hole

$$r_{\text{KN}\,\pm} = M \pm \sqrt{M^2 - a^2 - Q^2} \simeq r_{\text{K}\,\pm} \mp \frac{Q^2}{2\sqrt{M^2 - a^2}}, \quad (23)$$

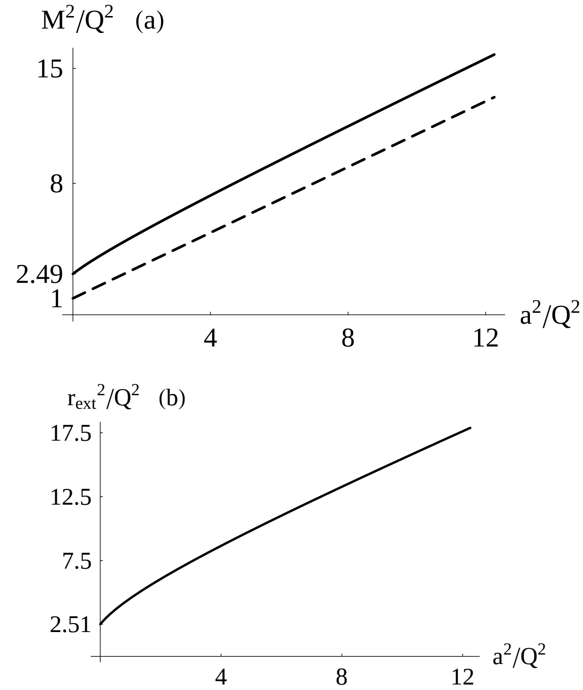

FIG. 1: (a): Using different horizontal and vertical scales, we show in the (a^2, M^2) plane the extremality condition. Continuous plot: Rotating regular black hole (10) with F given by (8). The black hole region is above this curved line. The curve itself represents an extremal black hole and the region beneath it represents regular non-black-hole solutions. Dashed plot: Rotating Kerr-Newman black hole. The Kerr-Newman black hole region is above this straight line of equation $M^2/Q^2 = a^2/Q^2 + 1$. Notice that the region between the two plots corresponds to $Q^2 < M^2$ which is within the black hole region for the Kerr-Newman solution but within the non-black-hole region ($\forall\, a^2 \geq 0$) for the rotating regular black hole (10). This is a parametric plot of $1/(2s)^2$ vs. u^2 (see Appendix). (b): The common radius ${r_{\text{ext}}}^2$ of the merging horizons vs. a^2. For $a^2 = 0$, we have ${r_{\text{ext}}}^2/Q^2 \simeq 2.51155$ yielding $r_{\text{ext}}/|Q| \simeq 1.58$ as found in [8]. This is a parametric plot of $t - 1$ vs. u^2 (see Appendix).

we obtain the order relations:

$$r_{\text{K}\,-} < r_{\text{KN}\,-} < r_- < r_+ < r_{\text{KN}\,+} < r_{\text{K}\,+}. \quad (24)$$

As far as the approximation $Q^2/M^2 \ll 1$ is valid, but this likely extends to all values of Q^2 within the limits of nonextremality, the horizons are ever closer than they are in Kerr or Kerr-Newman solutions.

The extremality condition and the common radius r_{ext} of the merging horizons are solutions to (21) along with $\partial\Delta/\partial r = 0$:

$$1 - \frac{M(r^2 + 4Q^2)r^2}{(r^2 + Q^2)^{5/2}} + \frac{2Q^4r^2}{(r^2 + Q^2)^3} = 0. \quad (25)$$

For $Q^2/M^2 \ll 1$, this leads to

$$M^2 \simeq a^2 + 4Q^2,\ r_{\text{ext}} \simeq M + \frac{3Q^2}{2M}. \quad (26)$$

[3] In [6], Ω was unintentionally misprinted as $g_{\theta\phi} = \Omega g_{\theta\theta}\sin^2\theta$. This is obviously a mistake since $g_{\theta\phi} \equiv 0$.

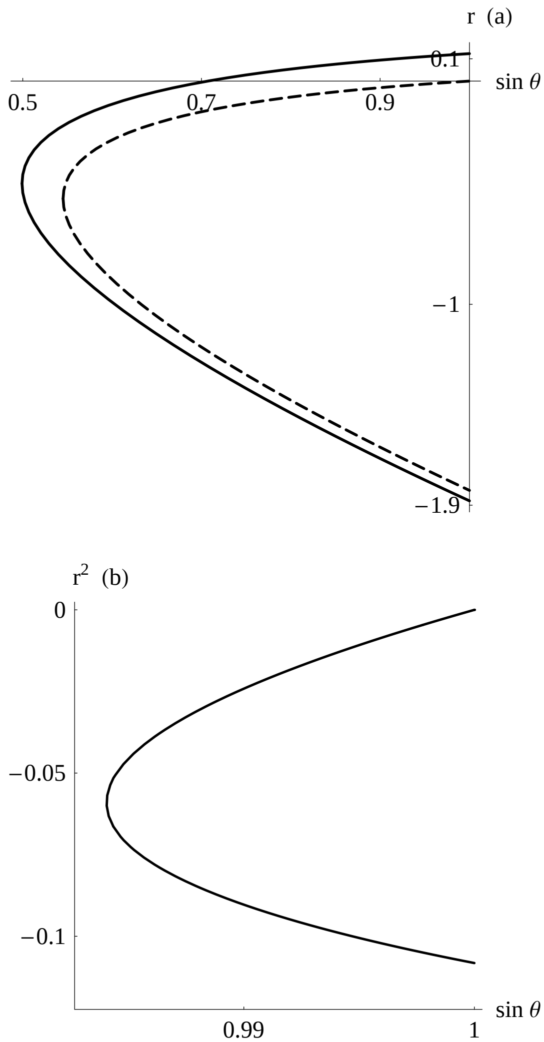

FIG. 2: For all the plots we took $M^2 = 16$, $Q^2 = 1$, and $a^2 = 1$ corresponding, according to Fig. 1 (a), to the black hole region for Kerr, Kerr-Newman, and the rotating regular black hole solution (10) with F given by (8). (a): Implicit plot of $\Sigma(\sin\theta, r) = 0$, where $\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$ and $\Delta = r^2F + a^2$, for Kerr black hole (dashed plot: $F = 1 - 2M/r$), Kerr-Newman black hole (continuous plot: $F = 1 - 2M/r + Q^2/r^2$), and the rotating regular black hole solution (10) with F given by (8) (the point $\sin\theta = 1$ and $r = 0$). Causality violations and CTCs occur on the right of each curve where $g_{\phi\phi} > 0$. The Kerr black hole has CTCs for $r < 0$ only while the Kerr-Newman one has CTCs for both signs of r. For the rotating regular black hole solution (10) with F given by (8) no causality violations or CTCs occur since $g_{\phi\phi} < 0$ [except at the point ($\sin\theta = 1$ and $r = 0$) where $\Sigma = 0$ and $g_{\phi\phi}$ is undefined]. (b): Implicit plot of $\Sigma(\sin\theta, r^2) = 0$ for the rotating regular black hole solution (10) with F given by (8). The plot confirms that solutions to $\Sigma(\sin\theta, r^2) = 0$ for $\sin\theta \neq 1$ are such that $r^2 < 0$.

The same values for an extremal Kerr-Newman black hole are $M^2 = a^2 + Q^2$, $r_{\text{KN ext}} = M$. The radius of the extremal rotating regular black hole is $3Q^2/(2M)$ larger than its Kerr-Newman counterpart.

For the same value of $M^2 - a^2$, ones sees that a Kerr-Newman black hole may cumulate three more levels of the electric charge $(M^2 - a^2)/4$ than a rotating regular one can do before the former becomes an extremal solution.

The latter conclusion extends to cases where the assumption $Q^2/M^2 \ll 1$ is not valid, as Fig. 1 (a) depicts. A consequence of that is to have no rotating and no static regular black holes for $Q^2 < M^2$ but only regular non-black-hole solutions for all values of $a^2 \geq 0$, as shown in Fig. 1 (a). It is clear from that figure that a horizontal line $M^2 = C$ where $Q^2 = 1 < 2.49 < C$ intersects the extremality condition curve, of the rotating regular black hole (10) with F given by (8), at some critical value ${a_c}^2$ above which the rotating solution is no longer a black hole. As C gets closer to 2.49, ${a_c}^2$ approaches zero, if rotation increases a bit ($a^2 \uparrow$) regular non-black-hole solutions become more favored than rotating black holes by nonlinear electrodynamics.

V. CONCLUSION

We have shown that metric (2.22) of Ref. [5] does not result from application of NJA and that there is no remedy to help overcome the situation but to postulate it independently of NJA. We have provided a method for generating, regular or singular, rotating black hole and non-black-hole solutions that is based partly on NJA but it avoids the complexification issues and employs physical arguments.

We have noticed that our rotating metric is much easier to handle than the one suggested in Ref. [5], we could provide simple treatments pertaining to the locations of the horizons and to the causality violations. It is easy to investigate analytically these issues using the metric (2.22) of Ref. [5].

We have concluded here and in [6, 7] that the rotating black hole and non-black-hole solutions (10) are regular on the ring $\rho^2 = 0$ but physical entities are undefined there.

Another interesting conclusion, confirmed in [14], is that the rotating regular black holes have much smaller electric charges and turn into regular non-black-hole solutions, for yet small charges, well before their Kerr-Newman counterparts become naked singularities. This remark extends most likely to all known regular black holes. The nonlinear electromagnetic field, due to the incursion of nonlinear electrodynamics in general relativity, are strong enough to help "vanishing" the horizons, for still small charges, well before their Kerr-Newman counterparts can do.

We have reached the conclusion that causality violations do not occur in the region $0 \leq r < r_-$ including small negative values of r for all rotating regular black holes. By symmetry of the static regular black holes, this conclusion extends down to $-r_-$.

The still remaining open issues are the determination of the electromagnetic potential and energetics of, as well as geodesic motion in, a rotating regular black hole.

Appendix: Extremality condition

We intend to find the extremality condition by solving both Eqs. (21) and (25). Let

$$u^2 \equiv a^2/Q^2,\ 2s \equiv |Q|/M, \tag{A.1}$$

$$x^2 \equiv {r_{\rm ext}}^2/Q^2,\ t = z^2 \equiv x^2 + 1 > 1, \tag{A.2}$$

the notations with x and s have been used in [8]. Eqs. (21) and (25), respectively

$$\frac{z^3}{z^2-1} - \frac{1}{s} + \frac{1}{z} + \frac{u^2z^3}{(z^2-1)^2} = 0, \tag{A.3}$$

$$1 - \frac{1}{2s}\frac{(z^2+3)(z^2-1)}{z^5} + \frac{2(z^2-1)}{z^6} = 0. \tag{A.4}$$

Solving (A.4) for s and using the result in (A.3) we arrive at

$$s = \frac{z(-3+2z^2+z^4)}{2(-2+2z^2+z^6)}, \tag{A.5}$$

$$1 - 3t - 3(u^2-2)t^2 - (5+u^2)t^3 + t^4 = 0. \tag{A.6}$$

Eq. (A.6) admits one and only one real root greater than 1 for all $u^2 \geq 0$: This is the root

$$t = \frac{5+u^2}{4} + \frac{\sqrt{W}}{2} + \frac{1}{2}\Big[Z + \frac{29+111u^2+27u^4+u^6}{4\sqrt{W}}\Big]^{1/2}, \tag{A.7}$$

where

$$\begin{aligned}
U &= \sqrt{428+828u^2+963u^4+16740u^6-1620u^8-108u^{10}},\\
V &= (36+27u^2+144u^4-18u^6+\sqrt{3}U)^{1/3},\\
W &= 2-u^2+3(u^2-2)+\frac{1}{4}(5+u^2)^2+\frac{V}{18^{1/3}}\\
&\quad+\frac{\left(\frac{2}{3}\right)^{1/3}(1-15u^2+3u^4)}{V},\\
Z &= u^2-2+3(u^2-2)+\frac{1}{2}(5+u^2)^2-\frac{V}{18^{1/3}}\\
&\quad-\frac{\left(\frac{2}{3}\right)^{1/3}(1-15u^2+3u^4)}{V}.
\end{aligned} \tag{A.8}$$

With the expression of t given by (A.7) and (A.8), the extremality condition reads substituting $[M^2/Q^2 = 1/(2s)^2]$ in (A.5):

$$\frac{M^2}{Q^2} = \frac{1}{t}\Big(\frac{t^3+2t-2}{t^2+2t-3}\Big)^2. \tag{A.9}$$

The plot of Fig. 1 (a) is a parametric plot of $1/(2s)^2$ vs. u^2 and that of Fig. 1 (b) is a parametric plot of $t-1$ vs. u^2.

---

[1] E.T. Newman and A.I. Janis, J. Math. Phys. **6**, 915 (1965).
[2] M. Demiański and E.T. Newman, Bull. Acad. Polon. Sci. Math. Astron. Phys. **14**, 653-657 (1966).
[3] F. Caravelli and L. Modesto, Class. Quantum Grav. **27**, 245022 (2010), arXiv:1006.0232.
[4] M. Azreg-Aïnou, Class. Quantum Grav. **28**, 148001 (2011), arXiv:1106.0970.
[5] B. Toshmatov, B. Ahmedov, A. Abdujabbarov, and Z. Stuchlík, Phys. Rev. D **89**, 104017 (2014), arXiv:1404.6443 [gr-qc].
[6] M. Azreg-Aïnou, Eur. Phys. J. C **74**, 2865 (2014), arXiv:1401.4292 [gr-qc].
[7] M. Azreg-Aïnou, Phys. Lett. B **730**, 95 (2014), arXiv:1401.0787.
[8] E. Ayón–Beato and A. García, Phys. Rev. Lett. **80**, 5056 (1998), arXiv:gr-qc/9911046.
[9] L. Modesto and P. Nicolini, Phys. Rev. D **82**, 104035 (2010), arXiv:1005.5605 [gr-qc].
[10] M. Gürses and F. Gürsey, J. Math. Phys. **16**, 2385 (1975).
[11] A. Burinskii, E. Elizalde, S.R. Hildebrandt, and G. Magli, Phys. Rev. D **65**, 064039 (2002), arXiv:gr-qc/0109085.
[12] L. Herrera and J. Jiménez, J. Math. Phys. **23**, 2339 (1982).
[13] S. Viaggiu, Int. J. Mod. Phys. D **15**, 1441 (2006), arXiv:gr-qc/0603036.
[14] L. Balart and E.C. Vagenas, Phys. Lett. B **730**, 14 (2014), arXiv:1401.2136 [gr-qc].