BTZ Black Hole Entropy in Loop Quantum Gravity and in Spin Foam Models

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March 13, 2013

Abstract

We present a comparison of the calculation of BTZ black hole entropy in loop quantum gravity and in spin foam models. We see that both give the same answer.

1 Introduction

In [1] a calculation of the BTZ black hole entropy from loop quantum gravity was obtained. The calculation follows similar steps to the calculation of the usual 4-dimensional case. The same calculation was done in [2] using spin foam models. The calculation is done by defining an expectation value using spin foam partition functions with observables.

Here we describe and compare both calculations see that both give the same answer.

We focus on the Euclidean version of the black hole. A three dimensional solution of Einstein's equations was introduced for the first time in [3]. Now it is known as the BTZ black hole. The three dimensional Euclidean solution to empty Einstein equations of general relativity with negative cosmological constant is given by the metric

$$ds^2 = \left(\frac{r^2}{\ell^2} - M\right) d\tau^2 + \left(\frac{r^2}{\ell^2} - M\right)^{-1} dr^2 + r^2 d\phi^2 \qquad (1)$$

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In [4], it is shown that by a change of coordinates, the solution can be written in the form

$$ds^2 = \frac{\ell}{z^2}(dx^2 + dy^2 + dz^2) \tag{2}$$

for $z > 0$. Immediately it can be recognised as the metric of the hyperbolic space H^3. Then after some isometric identifications the BTZ solution is in fact given by a fundamental region of the hyperbolic space. This region is a solid torus where the core of the torus is the black hole horizon $R = \ell\sqrt{M}$, and the rest of the torus is the outside of the black hole $R > \ell\sqrt{M}$.

According to Bekenstein-Hawking formula the leading term of the entropy of a black hole in three dimensions is given by

$$S \sim \frac{L}{4} \tag{3}$$

where L is the black hole horizon length. The entropy is believed to be related to the logarithm of the number of microstates.

2 The BTZ black hole entropy

In [1] the calculation of the entropy for the BTZ black hole is done in a similar way to the four dimensional case. The isolated three dimensional black hole horizon was introduced in [5]. The black hole horizon surface is thought as a circular boundary on a spacelike surface. It is also asummed that n spin network graph edges puncture the horizon(see Figure 1).

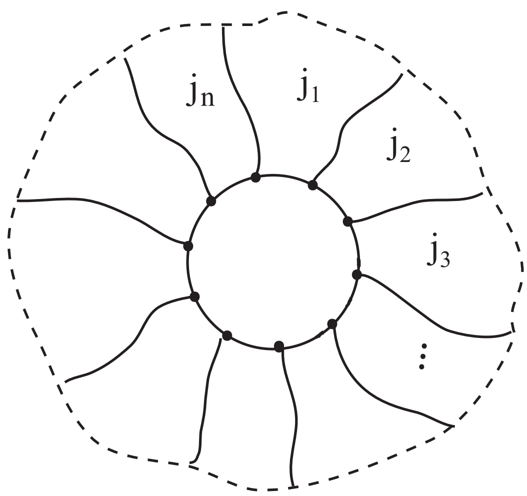

Figure 1: Spin network graph edges puncturing the horizon

The corresponding edges are labelled with irreducible representations of $SU(2)$. If $j_1, j_2, ..., j_n$ are the edge labels which cross the horizon. The length spectrum of three dimensional gravity was studied in [6]. The length of the horizon according to [1] is given by

$$L = 8\pi \ell_{Pl} \sum_{i=1}^{n} \sqrt{j_i(j_i+1)} \tag{4}$$

The number of states which give rise to the entropy is shown to be given by the dimension of the invariant tensor in the decomposition of the tensor product of irreducible representations of the quantum group version of $SU(2)$. This number is shown to be given approximately by

$$N = \frac{2}{k} \sum_{d=1}^{k} \sin^2\left(\frac{\pi}{k} d\right) \prod_{i=1}^{n} \frac{\sin(\frac{\pi}{k} d(2j_i+1))}{\sin(\frac{\pi}{k} d)} \tag{5}$$

After a kind of Wick rotation (which corresponds to making $k = i\lambda$) in order to have a negative cosmological constant, N is shown to be dominated by

$$N = \frac{2}{\lambda} \sinh^2(\pi) \prod_{i=1}^{n} \frac{\sinh(\pi(2j_i+1))}{\sinh(\pi)} \tag{6}$$

the entropy is given by the logarithm of the number of microstates. The authors from [1] claim that

$$S = \log(N) \sim \frac{L}{4\ell_{Pl}} \tag{7}$$

We now explain the entropy from the point of view of spin foam models, introduced in [2] and show that after a similar Wick rotation both calculations coincide.

The Euclidean BTZ black hole is topologically a solid torus. Consider a triangulation of the solid torus which contain interior edges, that is, the horizon is formed by edges (Figure 2).

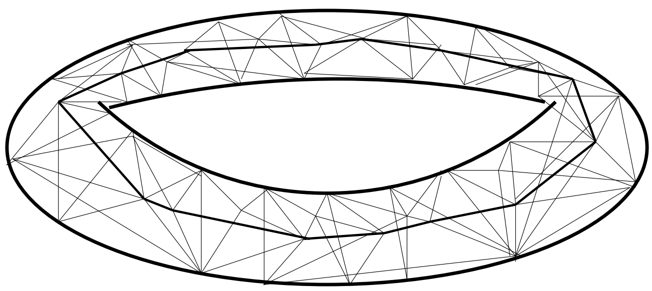

Figure 2: Triangulated BTZ Euclidean black hole

Let $Z(T^2, \mathcal{O})$ be the Turaev-Viro [7] partition function of the triangulated solid torus with the only difference that in the partition function we leave the labels at the horizon fixed. $Z(T^2, \mathcal{O})$ is therefore a function of these labels.[1]

We now think of the horizon as an observable and consider the expectation value of this observable defined by

$$W(T^2, \mathcal{O}) = \frac{Z(T^2, \mathcal{O})}{Z(T^2)} \tag{8}$$

where $Z(T^2)$ is the usual Turaev-Viro partition function of the solid torus.

The calculation of $W(T^2, \mathcal{O})$ was carried out in [2] and is computed by thinking of the blocks which form the triangulated horizon. Consider a particular triangulation which locally looks like in Figure 3 and where the horizon is triangulated with an even number of edges. Label the horizon edges by $i_1, j_1, \cdots, i_n, j_n$. Each pair of edges i_m, j_m belongs to a triangle which is labelled as $(i_m, j_m, \widehat{j_m})$. The edges labelled $\widehat{j_m}$ belong to the boundary of the solid torus.

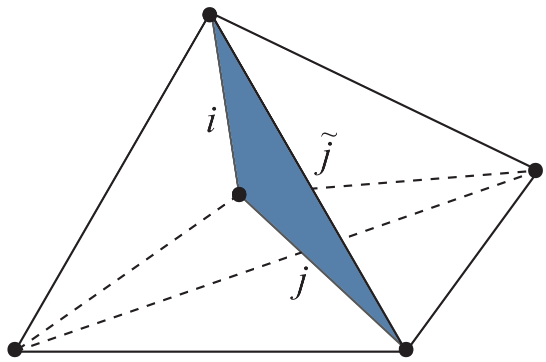

Figure 3: A block of the triangulated horizon where i and j belong to it and $\tilde{j}$ belongs to the boundary of the solid torus.

The expectation value of the horizon is given by

$$W(T^2, \mathcal{O}) = \prod_{m=1}^{n} \frac{N_{i_m, j_m, \widehat{j_m}}}{dim_q(\widehat{j_m})} dim_q(i_m) dim_q(j_m) \tag{9}$$

Recall that the factor $N_{i_m, j_m, \widehat{j_m}}$ is zero if the states are non admissible and 1 if states are admissible. We are considering admissible states since they will lead us to a non zero calculation.

We can rewrite the expectation value as[2]

[1] It must be understood that here we are dealing with the quantum group $SU_q(2)$. $q = e^{i\pi/r}$

[2] Here r is related to k of formula (5) by $k = 2(r-1)$

$$W(T^2,\mathcal{O}) = \prod_{m=1}^{n} \frac{(\frac{\frac{\sin\pi(2i_m+1)}{r}}{(\frac{\sin\pi}{r})})(\frac{\frac{\sin\pi(2j_m+1)}{r}}{(\frac{\sin\pi}{r})})}{\frac{(\frac{\sin\pi(2\widehat{j_m}+1)}{r})}{(\frac{\sin\pi}{r})}} \tag{10}$$

If we perform the same Wick rotation as in equation (6) and introduced in [1] we have that

$$W(T^2,\mathcal{O}) = \prod_{m=1}^{n} \frac{(\frac{\frac{\sinh\pi(2i_m+1)}{\lambda}}{(\frac{\sinh\pi}{\lambda})})(\frac{\frac{\sinh\pi(2j_m+1)}{\lambda}}{(\frac{\sinh\pi}{\lambda})})}{\frac{(\frac{\sinh\pi(2\widehat{j_m}+1)}{\lambda})}{(\frac{\sinh\pi}{\lambda})}} \tag{11}$$

We define the entropy to be given by

$$S = \log(W(T^2,\mathcal{O})) \tag{12}$$

Renaming the labels i_m and j_m by j_ℓ only, it can be seen that

$$S \simeq \sum_{j_\ell=1}^{2n} \log(\exp(\frac{\pi}{\lambda}2j_\ell)) - \sum_{m=1}^{n} \log\left(\frac{\sinh\pi(2\widehat{j_m}+1)/\lambda}{\sinh\pi/\lambda}\right) \tag{13}$$

and up to a factor we have that

$$S \simeq \log(N) - \sum_{m=1}^{n} \log\left(\frac{\sinh\pi(2\widehat{j_m}+1)/\lambda}{\sinh\pi/\lambda}\right) \tag{14}$$

If we consider the major contribution of the entropy, we have that our labels outside the horizon should vanish, which leads to $\widehat{j_m} = 0$, for all m.

Therefore the major contribution to the entropy is approximatelly given by

$$S \sim \log(N) \tag{15}$$

which up to a factor coincides with formula (7).

We have seen that loop quantum gravity and spin foam calculations lead to the same result. This really implies a very nice result since it suggests what is always expected; that loop quantum gravity and spin foams are truly the same theory.

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